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RAPID COMMUNICATIONS

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Critical states in a dissipative sandpile model

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A directed dissipative sandpile model is studied in two dimensions. Numerical results indicate that the long time steady states of this model are critical when grains are dropped only at the top, or everywhere. The critical behavior is mean-field-like. We discuss the role of infinite avalanches of dissipative models in periodic systems in determining the critical behavior of same models in open systems. [S1063-651X(99)50811-8]

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Spontaneous emergence of long range spatiotemporal correlations in driven dynamical systems without fine tuning of any control parameter, is the concept of self-organized criticality (SOC) [1–6]. Since its introduction in 1987 [1], the precise conditions that are necessary and sufficient for SOC have been subjected to intense scrutiny. The question that attracted much attention is, can one have criticality if there is a nonzero rate of bulk dissipation? While some works at the early stages [7] suggested that indeed, the conservation of the transported quantity in the dynamical rules is a necessity, the later works claimed a negative answer.

In this Rapid Communication, we study a directed dissipative sandpile model and our numerical results indicate that it is critical. We argue that a dissipative model may be critical provided the dissipation is not too strong, and conjecture a criterion to determine the critical behavior.

In the sandpile model of SOC, sand grains are locally injected and transported on an arbitrary lattice. Too many grains cannot be accommodated at any site. A site relaxes if the number of grains exceeds a certain cutoff and transfers the grains equally to the neighboring sites. This transfer process is conservative, where no grain is lost or created. At the critical state, cascades of relaxations follow due to single injection of grains, which are called avalanches. Grains, however, dissipate out of the system through the boundary, otherwise no steady state is possible. This is called the Abelian sandpile model (ASM) [1,4]. A globally driven conservative earthquake model is also similarly defined where energy is fed uniformly at all sites and transported [8]. This model reproduces power laws of energy release similar to the Gutenberg-Richter law [9].

There are some studies on the dissipative models also. Manna, Kiss, and Kertesz (MKK) studied a sandpile model where a grain can dissipate during a relaxing event, in a probabilistic manner. Numerical findings show that the system reaches a subcritical state with the characteristic sizes of the avalanches depending inversely on the probability of dissipation [10]. On the other hand, the dissipative ASM showed criticality with mean-field-like critical behavior [11]. A one-dimensional version of this model also showed critical behavior even with finite driving rate [12]. The Olami, Feder, and Christensen (OFC) model [13] studied the dissipative earthquake model, where dissipation is controlled by a parameter α. It is claimed that the OFC model is critical for $\alpha_c<\alpha<\alpha_o$, the conservative value of α being α_o, with the critical behavior depending on α [13–16]. The stochastic version of the OFC model, however, is shown to lose criticality for any $\alpha<\alpha_o$ [17].

In a conservative model of SOC the grains move a distance of the order of the system size L when started from the innermost region. This makes the average avalanche size

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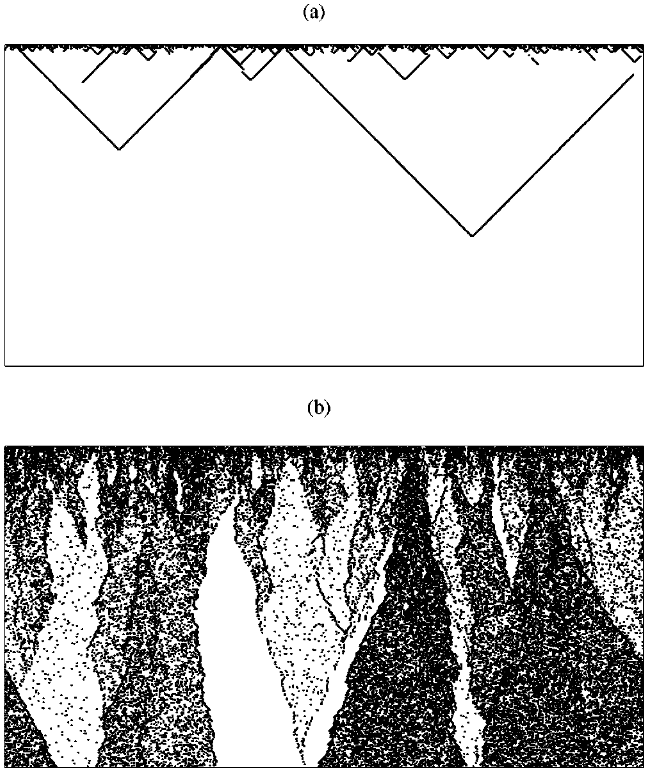

FIG. 1. Configurations of DDSM on a lattice of size $L=512$. (a) Avalanches created by dropping grains randomly only at the top row, orderly place grains along ''V'' shaped lines. (b) Grains are dropped randomly at all sites. Different densities in different regions indicate the age of the big avalnches passed through that region.

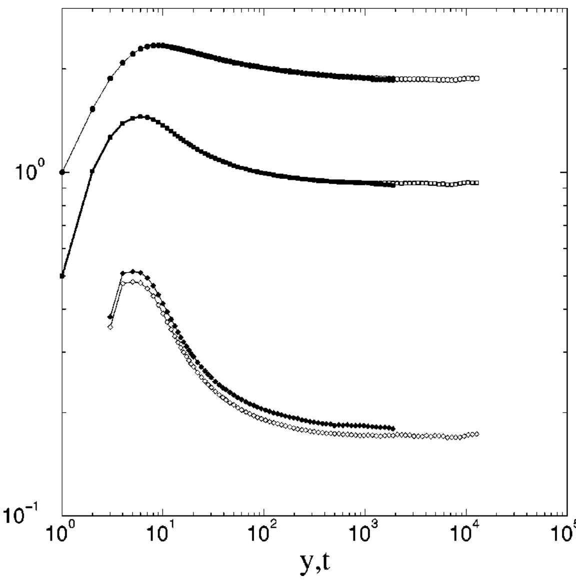

FIG. 2. Plots of $P(t)t^{\tau_t-1}$ (circles), $\rho(y)y^{\alpha}$ (squares), and $\sigma(y)y^{\beta}$ (diamonds) for system sizes $L=2048$ (closed symbols) and 13 000 (open symbols). Horizontal portions of the curves correspond to $\tau_t=2.027$, $\alpha=1.012$, and $\beta=1.184$.

grow as a power of L, so that an infinite system has a power law distribution of the avalanche sizes. In contrast, in a dissipative model, the grains dissipate at any distance within the system. If all grains do dissipate within certain cut-off distance, the average avalanche size would not have any dependence on L in the large limit. Therefore, for a dissipative model to be critical, only a fraction $f(L)$ of grains should dissipate from the bulk and the rest through the boundary, such that, $f(\infty)=Lt_{L\to\infty}f(L)\leqslant 1$. We present examples of both cases in the following.

On an oriented square lattice with extension L, sites are either vacant or occupied by single grains in the stable state. The system is periodic along the x direction and the y coordinate increases downward. Grains are dropped randomly. A toppling occurs only when the number of grains $h_i>1$, the site i is then vacated: $h_i\to 0$. The system has a preference along the y direction and the down-left and the down-right neighbors at the next row gets one grain each: $h_j\to h_j+1$. In a toppling with height 2, grain number is conserved, whereas, in a toppling with height 3, one grain dissipates from the system. Unlike the directed Abelian sandpile model (DASM) [18], our model is non-Abelian since sites are vacated in a toppling and we call it as the ''directed dissipative sandpile model'' (DDSM).

In a parallel dynamics all toppling sites reside in a single row on a contiguous toppling line (TL). It has a variable length since the fluctuations take place only at the two ends. If the TL has a length ℓ at time t, it would have the length at least $\ell-1$ at time $(t+1)$, since all inner $\ell-1$ sites will get two grains each and will topple again. If either or both neighbors of the end sites are occupied, the TL collects the grains into it, and grows in length to ℓ, or $\ell+1$. However, if these neighbors are vacant, TL fills them and shrinks in length. Therefore, the two ends of the TL move in principle like two annihilating random walks starting from the same point. The avalanche terminates when they meet and annihilate each other [18].

Grains are randomly dropped in two ways: In case A, they are dropped only on the top row at $y=1$, and in case B, they are dropped everywhere. Grains dissipate through the boundary at $y=L$. First, we consider case A and a stable configuration is shown in Fig. 1(a). Grains are marked by black dots, where as vacant sites are made blank. Lines of grains in the shape of ''V'' are mostly observed. This is because, due to the bulk dissipation, the density is so low that the TL moves almost in a deterministic manner. A ''V'' is formed by the movement of a TL through a vacant region. In this case the TL uniformly shrinks, leaving behind a trail of two converging lines of occupied sites at the two ends. However, a TL may also propagate in a ''Λ'' between two Vs. In that case it uniformly grows in length, deletes two sides of two Vs up to their lowest points and then starts shrinking, producing two converging lines which finally make a bigger V. In this way bigger V shapes are generated at the expense of smaller Vs, which finally reaches the boundary at the bottom and dissipates. Such almost deterministic dynamics makes avalanches of rectangular shape in general, but mostly they are squares.

An avalanche deletes all occupied sites through which it passes. No dissipation occurs in the first two rows where the average density is 1/2, as in DASM [18]. It then decreases with y as a power law: $\rho(y)\sim y^{-\alpha}$. A system of size $L=13\,000$ is simulated by dropping 2×10^9 grains. We plot $\rho(y)y^{1.012}$ with y on a double logarithmic scale, the curve is horizontal for the large y values, giving $\alpha=1.012\pm0.030$ (Fig. 2).

Avalanche size s is the number of sites toppled in an

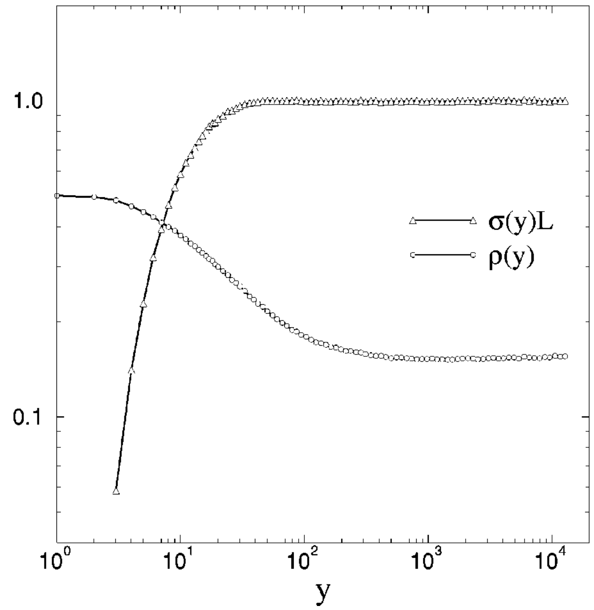

FIG. 3. Saturation of the density $\rho(y)$ and the fraction of bulk dissipation $\sigma(y)$ in a system of size $L=13\,000$.

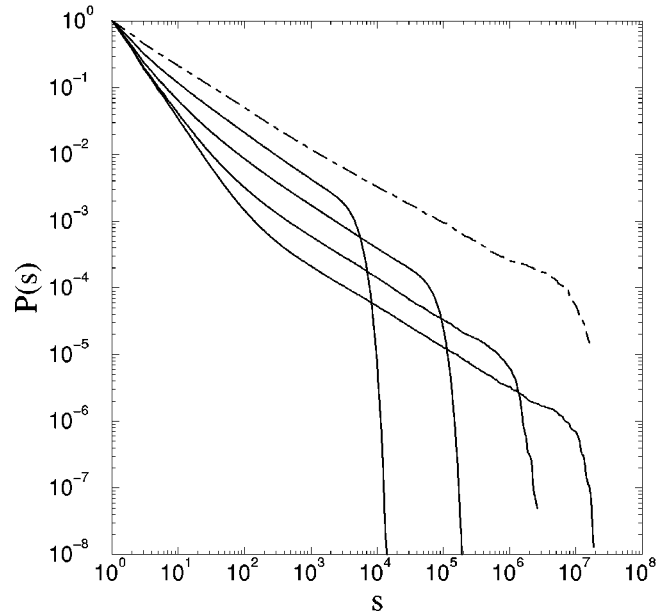

FIG. 4. The avalanche size distribution for case *B* of DDSM. The dot-dashed curve is for the high density region. The four curves with solid lines are for avalanches of the whole system for the system sizes $L=256$, 1024, 4096, and 13 000 (from top to bottom).

avalanche. Simulation results indicate that the cumulative probability distribution of s follows a power law: $P(s) \sim s^{1-\tau_s}$, with $\tau_s=1.52\pm0.03$. The lifetime t of an avalanche is its vertical extension along the preferred direction and also follows similar power law: $P(t)\sim t^{1-\tau_t}$, with $\tau_t=2.027 \pm 0.030$ (Fig. 2). The average avalanche size $\langle s(t)\rangle$ varies with lifetime t as $\langle s(t)\rangle \sim t^{\gamma_{st}}$, with $\gamma_{st}=2.01\pm0.03$. Since s,t are two measures of the same random avalanche cluster, they are necessarily dependent, and are related by the scaling relation: $\gamma_{st}=(\tau_t-1)/(\tau_s-1)$.

We explain these results in the following way. It is reasonable to assume that most of the avalanches are of rectangular shapes, which implies that $\gamma_{st}=2$. Now, if the TL has a width $w(t')$ at the intermediate time t', then $2w(t')$ grains cross that row $y=t'$. The dissipation flux per grain can be divided into ''bulk-flux'' and ''boundary-flux.'' All grains crossed by the TL, except at its two ends, dissipate. Therefore, the density and the system size L control the share between the bulk and the boundary fluxes. The constant average boundary-flux through the row at y is $\langle w(y)\rangle y^{1-\tau_t}$, which gives $\langle w(y)\rangle \sim y^{\tau_t-1}$. But, since the average avalanche size of lifetime t is $\langle s(t)\rangle = \int_0^t w(t')dt' = t^{\tau_t}$, we get $\gamma_{st}=\tau_t=2$ and $\tau_s=3/2$. We numerically check the relation $\langle w(y)\rangle = ky$ and find a nice straight line with slope $k = 0.312\pm0.001$ and the correlation coefficient 0.999.

Since the density of the system decreases with increasing y, we expect that the average dissipation also should decrease with increasing y. The fraction $\sigma(y)$ of total number of grains dissipated in the yth row varies as $\sigma(y)\sim y^{-\beta}$, where $\beta=1.184\pm0.030$ (Fig. 2). Therefore, the bulk-flux $f(L)$ should vary as $f(L)=f(\infty)-CL^{-x}$, with $x=\beta-1$. The exponent x is estimated independently by plotting $f(L)$ versus L^{-x} for different x values, and the best value obtained is $x = 0.17\pm0.03$ and $f(\infty)=0.634\pm0.010$.

Now we consider case *B* [Fig. 1(b)]. The local density fluctuates widely since an avalanche sweeps the region it passes and the local density in this region restarts to grow from scratch. Since the grains are dropped everywhere uniformly and the bulk dissipation depends on the density, we expect that there should be a saturation region where the average density is constant. The density $\rho(y)$ decreases from 1/2 and then saturates to a constant value 0.1543 ± 0.0010 around $y_c\approx100$. The rate of dissipation $\sigma(y)$, initially increases but finally saturates to the uniform dissipation limit: $\sigma(y)=C/L$, with $C=1.01$ (Fig. 3). The bulk-flux $f(L)$ asymptotically reaches to $f(\infty)=1$ as $1/L$.

It turned out that the system has two regions. The high density region extends from the top to y_c and the saturation region from y_c to L. We separately collect the distribution data for the avalanches originated in these two regions. For the high density region, the $\tau_s\approx1.5$ and $\tau_t\approx2.0$ are obtained as in case *A* and $\langle s(L)\rangle \sim L$ and $\langle t(L)\rangle \sim \log L$ are observed. Linearity in $\langle w(t)\rangle = k_1 t$ is still obeyed with $k_1=0.1$ and $\gamma_{st}\approx2$ is obtained again. However, for the saturation region, plots of the distribution data showed two regions: an initial high slope $\tau_s^s\approx2.5$ for the small s values, followed by a slope

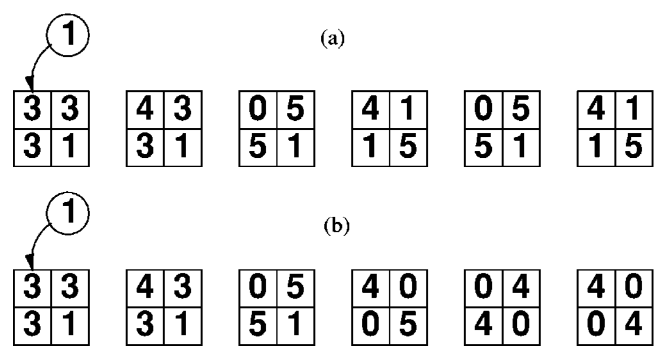

FIG. 5. (a) ASM on a periodic 2×2 lattice, which leads to a periodic infinite avalanche. (b) Dissipative ASM on a similar periodic lattice also leads to the periodic infinite avalanche.

$\tau_s^l \approx 1.5$ for the large s values (see Fig. 4). We explain that the large value of τ_s^s is due to those small avalanches, which grow on an empty region swept out by a previous large avalanche. When this region reaches the steady state, the avalanches get the usual exponent $\tau_s^l = 1.5$ for large s values. We see that both $\langle s(L) \rangle$ and $\langle t(L) \rangle$ have constant values independent of L. The total distribution has the behavior of the saturated regions, since the avalanches generated in this region have larger weights.

We now look into the effect of the boundary on dissipative models in more detail. In a conservative sandpile model with periodic boundary condition, the total mass of the system grows up indefinitely. Very soon, an ''infinite avalanche'' starts which never terminates. For ASM on a periodic square lattice, the same height configuration repeats at certain interval, toppling all sites exactly once. The period is of the order of L and is dependent on the initial configuration. We show such a 2×2 system in Fig. 5(a). Next we consider the dissipative ASM [11] on the same lattice. After some initial dissipation, this model also creates a periodic infinite avalanche which is dissipationless [Fig. 5(b)]. We now test our DDSM on a periodic system, by making the y direction also periodic. We observe again dissipationless infinite avalanches in both cases A and B. A TL in the form of a ring moves indefinitely with uniform speed on the empty torus.

An infinite avalanche has to be dissipationless after some time, otherwise it will make the whole system empty. If a dissipative model in a periodic system has no infinite avalanche, it indicates that even the largest avalanche is not as big as the system. Therefore for the same dissipative model in the open system, the boundary should have no effect on the avalanche sizes, leading to subcritical states. We conjecture that: A dissipative model will not show self-organized criticality if the same model on a periodic system has no infinite avalanches.

To verify this conjecture, we check some examples. The probabilistic dissipation model [10], the stochastic OFC [17] model, and the dissipative two-state sandpile model [19] are all noncritical on open boundary systems and do not have infinite avalanches on the periodic systems. However, the random creation-dissipation model in [10], the dissipative ASM [11], and the cases of DDSM as described in this paper, lead to SOC states with open boundary and also have periodic avalanches on the periodic systems. Finally, we check that the deterministic OFC model [13] also does not produce any infinite avalanche on the periodic system for any $\alpha < 1/4$. Therefore, according to our conjecture, deterministic OFC model is not critical, which is against the general belief. Recently, it has been claimed that the deterministic OFC model is critical in the conservative regime only [20].

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